PERGAMON

International Journal of Solids and Structures 40 (2003) 4353–4378

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Finite strain—isotropic hyperelasticity

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Received 15 March 2003

Abstract

This paper presents a strain energy density for isotropic hyperelastic materials. The strain energy density is decomposed into a compressible and incompressible component. The incompressible component is the same as the generalized Mooney expression while the compressible component is shown to be a function of the volume invariant J only. The strain energy density proposed is used to investigate problems involving incompressible isotropic materials such as rubber under homogeneous strain, compressible isotropic materials under high hydrostatic pressure and volume change under uniaxial tension. Comparison with experimental data is good. The formulation is also used to derive a strain energy density expression for compressible isotropic neo-Hookean materials. The constitutive relationship for the second Piola–Kirchhoff stress tensor and its physical counterpart, involves the contravariant Almansi strain tensor. The stress stretch relationship comprises of a component associated with volume constrained distortion and a hydrostatic pressure which results in volumetric dilation. An important property of this constitutive relationship is that the hydrostatic pressure component of the stress vector which is associated with volumetric dilation will have no shear component on any surface in any configuration. This same property is not true for a neo-Hookean Green's strain–second Piola–Kirchhoff stress tensor formulation.

Keywords: Finite strain; Elasticity; Beam theory; Hyperelasticity; Rubber

1. Introduction

This paper is the first of two papers dealing with finite strain elasticity analysis of beams. In an attempt to review what is appropriate for the non-linear analysis of beams, a general strain energy density for isotropic hyperelastic materials is first developed. Based on several postulates as to the conditions which must be met by the strain energy density an expression is proposed consisting of incompressible and compressible components. The incompressibility component is the "general" Mooney (1940) expression for higher order elasticity and satisfies the Valanis–Landel hypothesis. The compressibility component of the strain energy density for an isotropic material is shown to be a function of the volume invariant J only, and is the strain energy produced by the application of a hydrostatic pressure. The expression proposed is a generalisation

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doi:10.1016/S0020-7683(03)00217-8

of the Simo and Pister (1984) proposal for a neo-Hookean material. The compressibility component is associated with volumetric dilation while the incompressibility component is associated with volume constant distortion. The compressibility component of the strain energy density leads to a hydrostatic pressure component of the stress vector which must have no shear component on any surface of the material in any configuration. Any shear on any surface will not be a function of the material properties which produce volumetric dilation. Several examples are detailed involving tests on rubber at large deformations under homogeneous strain, compressible materials under large hydrostatic pressure and measurements of volume changes under uniaxial tension.

The strain energy density for a compressible neo-Hookean material is derived which is the same as that proposed by Simo and Pister (1984). The constitutive law for the second Piola–Kirchhoff stress tensor for a neo-Hookean material involves the contravariant Almansi strain tensor and the volumetric invariant J. The basis of many non-linear and stability analyses of structures is a neo-Hookean constitutive relationship between Green's strain tensor and its conjugate stress, the second Piola–Kirchhoff stress tensor. This is somewhat in doubt because of the Engesser/Haringx controversy (see Bazant, 1971; Bazant and Cedolin, 1991; Bazant, 2003; Attard, 2003) and several other reasons detailed below. The strain energy density for a Hookean Green's strain tensor–second Piola–Kirchhoff stress tensor formulation would be $aI_\varepsilon^2 + bII_\varepsilon$ where a and b are material constants and I_ε and II_ε are the first and second strain invariants. When all the principal Green's strains are collapsed to a singularity they all have a value of $-1/2$ and the strain invariants are both non-zero. This implies that the strain energy density and the associated stresses required to collapse a material to a singularity would have a finite value. This seems physically objectionable. In this paper, it is shown that when a neo-Hookean constitutive relationship between Green's strain tensor and the second Piola–Kirchhoff stress tensor is used and the stresses acting on a surface of a deformed body are divided into components normal and tangential to the surface, the resulting expression for the tangential shear stress is a function of the material parameters associated with volumetric dilation. This would also seem objectionable.

Fig. 1 shows a two-dimensional view of a deformed element with stresses on the vertical faces divided into a normal component and a tangential shear component. It is shown in this paper, that the normal stress is a function of the normal component of stretch and the volumetric dilation while the shear stress is only a function of the shear component of stretch. The shear stress taken orthogonal to the normal stress

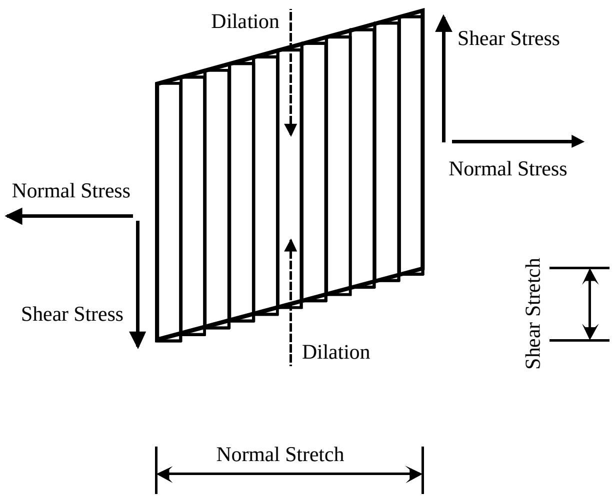

Fig. 1. Two-dimensional element.

should not be a function of the volumetric dilation. The example of simple shear is looked at in some detail to show the different stresses which result from the proposed constitutive law as compared to those which result from assuming a neo-Hookean constitutive relationship between Green's strain tensor and the second Piola–Kirchhoff stress tensor.

2. Preliminaries—kinematics of a continuum

Consider a continuum of material at rest in an undeformed state. Particles within the continuum can be thought of as forming natural lines or chains of particles called material lines (Wempner, 1981). If the material is deformed these chains of particles move in such a way that particles remain on the same material line, that is material lines always remain intact. A particle within this continuum is denoted by point P with coordinates $x^i(\theta^1, \theta^2, \theta^3)$, $i = 1, 2, 3$, with respect to a fixed three-dimensional Cartesian coordinate system, assumed to be a function of general coordinates θ^i, $i = 1, 2, 3$. The θ^i can be viewed as curvilinear or intrinsic coordinates along the material lines. The convention due to Einstein is adopted where a repeated index such as in $p_i v^i$ is used to imply summation. The repeated index is called a dummy variable as it can be changed to any symbol without altering the meaning of the summation. A bracketed index indicates suppression of the summation convention, e.g. $x_{(ii)}$.

The position vector $\mathbf{R}$ (the bold style $\mathbf{R}$ is used to distinguish a vector while vector components will be written in italics e.g. x^i) of the particle P in the undeformed state is given by

$$\mathbf{R} = \mathbf{i}_i x^i(\theta^1, \theta^2, \theta^3) \tag{1}$$

where $\mathbf{i}_i$ are unit Cartesian vectors (refer to Fig. 2). To examine deformations of the continuum, we first define a differential line element vector $\mathrm{d}\mathbf{s}$ at particle P in the undeformed state, given by

$$\mathrm{d}\mathbf{s} = \frac{\partial \mathbf{R}}{\partial \theta^i}\mathrm{d}\theta^i = \frac{\partial x^j}{\partial \theta^i}\mathbf{i}_j\,\mathrm{d}\theta^i = x^j_{,i}\,\mathbf{i}_j\,\mathrm{d}\theta^i = \mathbf{g}_i\,\mathrm{d}\theta^i \tag{2}$$

with respect to covariant tangent base vectors $\mathbf{g}_i$. The comma notation indicates differentiation with respect to θ^i. The tangent base vectors are so-called because they are tangential to the natural material lines. These base vectors are "not" necessarily unit vectors and may not be dimensionless. The contravariant base

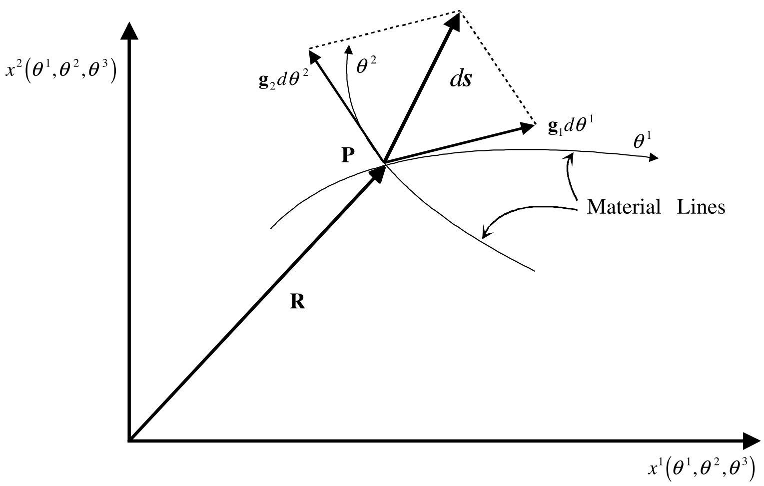

Fig. 2. Undeformed position.

vectors are normal to the material lines and are sometimes referred to as reciprocal base vectors. The scalar product of covariant and contravariant base vectors is the kronecker delta δ_i^j,

$$\mathbf{g}_i \cdot \mathbf{g}^j = \mathbf{g}^j \cdot \mathbf{g}_i = \delta_i^j \tag{3}$$

3. The metric tensor

The square of the length of the differential line element is called the metric and is calculated from:

$$\mathrm{d}\mathbf{s} \cdot \mathrm{d}\mathbf{s} = \mathbf{g}_i \,\mathrm{d}\theta^i \cdot \mathbf{g}_j \,\mathrm{d}\theta^j = g_{ij} \,\mathrm{d}\theta^i \,\mathrm{d}\theta^j \tag{4}$$

The term g_{ij} is the covariant metric tensor in the undeformed coordinate system. It follows from Eq. (4) that the metric tensor is symmetric. The determinant of the metric tensor g_{ij} is written as

$$g = \det(g_{ij}) \tag{5}$$

If the determinant of the metric tensor is strictly positive, the space is called a Riemannian space. The contravariant metric tensor g^{ij} can be derived in a similar fashion. The relationship between the covariant and contravariant metric tensors is

$$g^{ij} g_{jk} = \delta_k^i \tag{6}$$

The covariant and contravariant metric tensors also have the important property (see Green and Zerna, 1968 or Renton, 1987).

$$\frac{\partial g}{\partial g_{ij}} + \frac{\partial g}{\partial g_{ji}} = 2 g g^{ij} \tag{7}$$

4. Stretch

Consider a particle P within a continuum that moves to a new position $\hat{P}$ with the position vector $\hat{\mathbf{R}}$ as shown in Fig. 3. The new position is assumed to be a function of the coordinates θ^i and are therefore said to be convected coordinates (Flügge, 1972). The new position vector is given by

$$\hat{\mathbf{R}} = \mathbf{R} + \mathbf{u} \tag{8}$$

in which $\mathbf{u}$ are displacements assumed to be smooth and differentiable. A differential line element vector $\mathrm{d}\hat{\mathbf{s}}$ at particle $\hat{\mathbf{P}}$ is given by

$$\mathrm{d}\hat{\mathbf{s}} = \frac{\partial \hat{\mathbf{R}}}{\partial \theta^i} \mathrm{d}\theta^i = G_i^j \mathbf{g}_j \,\mathrm{d}\theta^i = \hat{\mathbf{g}}_i \,\mathrm{d}\theta^i \tag{9}$$

with $\hat{\mathbf{g}}_i$ the covariant tangent base vectors, G_j^i the covariant deformation gradient tensor (Fig. 3). The associated inverse deformation gradient tensor is denoted by $\overline{G}_j^i$ and satisfies the following:

$$\overline{G}_k^i G_j^k = G_k^i \overline{G}_j^k = \delta_j^i \tag{10}$$

The base vectors in the undeformed and deformed state can be related by

$$\hat{\mathbf{g}}^i = \overline{G}_j^i \mathbf{g}^j \quad \hat{\mathbf{g}}_i = G_i^j \mathbf{g}_j \quad \mathbf{g}_i = \overline{G}_i^j \hat{\mathbf{g}}_j \quad \mathbf{g}^i = G_j^i \hat{\mathbf{g}}^j \tag{11}$$

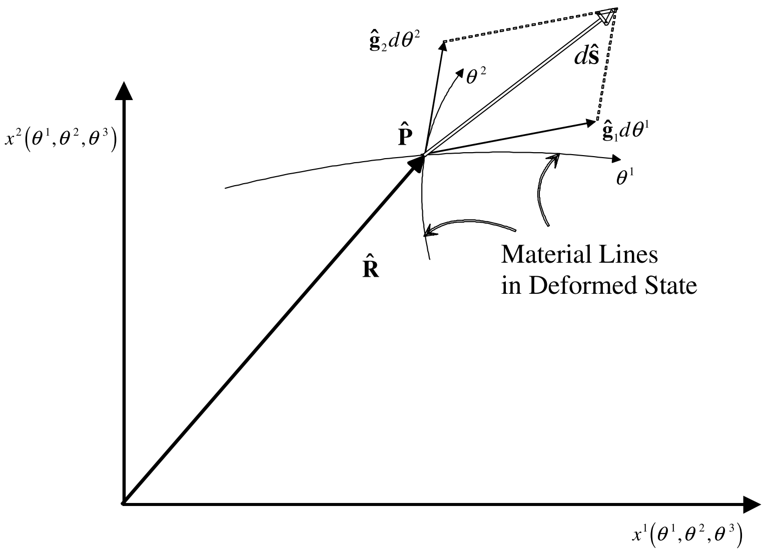

Fig. 3. Deformed position.

The differential line element at the particle P, can be visualized as the diagonal of a parallelepiped with sides corresponding to the vectors $\mathbf{g}_1\,\mathrm{d}\theta^1$, $\mathbf{g}_2\,\mathrm{d}\theta^2$ and $\mathbf{g}_3\,\mathrm{d}\theta^3$. The ratio of the change in lengths of the sides of this parallelepiped to their initial length as P moves to $\hat{P}$, are measures of relative elongations. Consider the material arc at P corresponding to the vector $\mathbf{g}_1\,\mathrm{d}\theta^1$, after deformation the length of this fibre becomes

$$|\hat{\mathbf{g}}_1\,\mathrm{d}\theta^1| = \lambda_1|\mathbf{g}_1\,\mathrm{d}\theta^1| \tag{12}$$

where λ_i are the relative stretches associated with the ith covariant material arc and are therefore defined by

$$\lambda_i = \sqrt{\frac{\hat{g}_{(ii)}}{g_{(ii)}}} \tag{13}$$

The quantity λ_i is not a tensor and represents the relative stretch of the material arc $\mathbf{g}_1\,\mathrm{d}\theta^1$. Since a stretch of "0" would indicate a singularity, the stretch must be positive for all deformations.

$$\lambda_i > 0 \tag{14}$$

If the material lines are orthogonal both in the initial and final configuration of the continuum, the covariant and contravariant metric tensor are related to the relative stretches by

$$\hat{g}_{(ii)} = (\lambda_i)^2 \quad \hat{g}^{(ii)} = \frac{1}{(\lambda_i)^2} \tag{15}$$

5. Strain tensor

The square of the length of the differential line element $\mathrm{d}\hat{\mathbf{s}}$ at particle $\hat{P}$ is given by

$$\mathrm{d}\hat{\mathbf{s}}\cdot\mathrm{d}\hat{\mathbf{s}} = \hat{\mathbf{g}}_i\,\mathrm{d}\theta^i\cdot\hat{\mathbf{g}}_j\,\mathrm{d}\theta^j = \hat{\mathbf{g}}_i\cdot\hat{\mathbf{g}}_j\,\mathrm{d}\theta^i\,\mathrm{d}\theta^j = \hat{g}_{ij}\,\mathrm{d}\theta^i\,\mathrm{d}\theta^j \tag{16}$$

where $\hat{g}_{ij}$ is the metric tensor in the deformed coordinate system. The change in length of the square of the differential line element can be used to characterize the deformation and to define a strain tensor γ_{ij}, thus

$$\mathrm{d}\hat{\mathbf{s}} \cdot \mathrm{d}\hat{\mathbf{s}} - \mathrm{d}\mathbf{s} \cdot \mathrm{d}\mathbf{s} = (\hat{g}_{ij} - g_{ij})\,\mathrm{d}\theta^i\,\mathrm{d}\theta^j = 2\gamma_{ij}\,\mathrm{d}\theta^i\,\mathrm{d}\theta^j \tag{17}$$

The components of the strain tensor are not necessarily dimensionless because the initial base vectors are not unit vectors. The dimensionless counterpart to the strain tensor, the physical Green's Lagrangian strain ε_{ij} is given by

$$\varepsilon_{ij} = \frac{\gamma_{ij}}{\sqrt{g_{(ii)}}\sqrt{g_{(jj)}}} \tag{18}$$

For an initial Cartesian coordinate system, the physical Green's Lagrangian strain is equal to the strain tensor, γ_{ij}.

An alternate definition for a strain tensor can be derived in Eulerian coordinates that is with respect to coordinates in the deformed state rather than the undeformed. Using the deformation gradient tensor defined in Eq. (9), the differential line elements $\mathrm{d}\hat{\mathbf{s}}$ and $\mathrm{d}\mathbf{s}$ can be written with respect to new coordinates $\mathrm{d}\bar{\theta}^i$ aligned with the initial tangent base vectors. That is

$$\begin{aligned} \mathrm{d}\hat{\mathbf{s}} &= \hat{\mathbf{g}}_i\,\mathrm{d}\theta^i = \mathbf{g}_j\,G_i^j\,\mathrm{d}\theta^i = \mathbf{g}_j\,\mathrm{d}\bar{\theta}^j \\ \mathrm{d}\mathbf{s} &= \mathbf{g}_i\,\mathrm{d}\theta^i = \mathbf{g}_j\,\overline{G}_i^j\mathrm{d}\bar{\theta}^i \end{aligned} \tag{19}$$

The change in length of the square of the differential line element is then

$$\mathrm{d}\hat{\mathbf{s}} \cdot \mathrm{d}\hat{\mathbf{s}} - \mathrm{d}\mathbf{s} \cdot \mathrm{d}\mathbf{s} = (g_{ij} - \overline{G}_i^k\overline{G}_j^l g_{kl})\,\mathrm{d}\bar{\theta}^i\mathrm{d}\bar{\theta}^j = 2\bar{\gamma}_{ij}\mathrm{d}\bar{\theta}^i\mathrm{d}\bar{\theta}^j \tag{20}$$

where $\bar{\gamma}_{ij}$ is the Almansi strain tensor. The relationship between the two strain measures is given by

$$\bar{\gamma}_{ij} = \overline{G}_i^k\overline{G}_j^l\gamma_{kl} \tag{21}$$

A second Almansi strain tensor can also be defined with respect to covariant coordinates defined by $\mathrm{d}\bar{\theta}_i = G_i^j\,\mathrm{d}\theta_j$ aligned with the contravariant tangent base vectors in the deformed state. The differential line elements $\mathrm{d}\hat{\mathbf{s}}$ and $\mathrm{d}\mathbf{s}$ can be thus written in the form:

$$\mathrm{d}\hat{\mathbf{s}} = \hat{\mathbf{g}}_i\,\mathrm{d}\theta^i = \mathbf{g}^i\,\mathrm{d}\bar{\theta}_i \quad \mathrm{d}\mathbf{s} = \mathbf{g}_i\,\mathrm{d}\theta^i = \hat{\mathbf{g}}^i\,\mathrm{d}\bar{\theta}_i \tag{22}$$

The change in length of the square of the differential line element is then

$$\mathrm{d}\hat{\mathbf{s}} \cdot \mathrm{d}\hat{\mathbf{s}} - \mathrm{d}\mathbf{s} \cdot \mathrm{d}\mathbf{s} = (g^{ij} - \hat{g}^{ij})\,\mathrm{d}\bar{\theta}_i\,\mathrm{d}\bar{\theta}_j = 2\bar{\gamma}^{ij}\,\mathrm{d}\bar{\theta}_i\,\mathrm{d}\bar{\theta}_j \tag{23}$$

where $\bar{\gamma}^{ij}$ is the contravariant Almansi strain tensor. The contravariant Almansi strain tensor is related to the Doyle–Ericksen strain measure with $m = -2$ (refer to Ogden, 1997, p. 119).

6. Invariants

There are many tensor invariants which can be written in terms of the metric tensor in the undeformed and deformed state, as well as the relative stretches. The most common quoted triad of invariants (I_λ, II_λ, III_λ) are:

$$I_\lambda = g^{ij}\hat{g}_{ij} = (\lambda_{p1})^2 + (\lambda_{p2})^2 + (\lambda_{p3})^2 = (\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2 \tag{24}$$

$$II_\lambda = g_{ij}\hat{g}^{ij}III_\lambda = \tfrac{1}{2}(I_\lambda^2 - g^{ki}g^{lj}\hat{g}_{kj}\hat{g}_{li}) = (\lambda_{p1}\lambda_{p2})^2 + (\lambda_{p1}\lambda_{p3})^2 + (\lambda_{p2}\lambda_{p3})^2$$
$$= (\lambda_1\lambda_2)^2 \sin^2\hat{\phi}_{12} + (\lambda_1\lambda_3)^2\sin^2\hat{\phi}_{13} + (\lambda_2\lambda_3)^2\sin^2\hat{\phi}_{23} \quad (25)$$

$$J = (III_\lambda)^{1/2} = \sqrt{\frac{\det(\hat{g}_{ij})}{\det(g_{ij})}} = \frac{\sqrt{\hat{g}}}{\sqrt{g}} = \lambda_{p1}\lambda_{p2}\lambda_{p3}$$
$$= \lambda_1\lambda_2\lambda_3(1 + 2\cos\hat{\phi}_{12}\cos\hat{\phi}_{13}\cos\hat{\phi}_{23} - \cos^2\hat{\phi}_{12} - \cos^2\hat{\phi}_{13} - \cos^2\hat{\phi}_{23})^{1/2} \quad (26)$$

where λ_{p1}, λ_{p2} and λ_{p3} are the principal stretches, λ_1, λ_2 and λ_3 are the stretches when the initial coordinate system is Cartesian and $\hat{\phi}_{ij}$ are the angles between the ith and jth tangent base vectors in the deformed state. The invariant I_λ represents the sum of the squares of relative ratios of the three distinct sides of the deformed parallelepiped, II_λ the sum of the relative ratios of the squares of the three distinct surface areas of the deformed parallelepiped and III_λ the relative ratio of the square of the volume of the deformed parallelepiped.

Another set of invariants which will prove useful later are defined by

$$L_1 = I_\lambda = g^{ij}\hat{g}_{ji} = (\lambda_{p1})^2 + (\lambda_{p2})^2 + (\lambda_{p3})^2$$
$$L^1 = I_{\lambda^{-1}} = g_{ij}\hat{g}^{ji} = \frac{1}{(\lambda_{p1})^2} + \frac{1}{(\lambda_{p2})^2} + \frac{1}{(\lambda_{p3})^2} \quad (27)$$

$$L_2 = g^{ki}g^{lj}\hat{g}_{jk}\hat{g}_{il} = (\lambda_{p1})^4 + (\lambda_{p2})^4 + (\lambda_{p3})^4$$
$$L^2 = g_{ki}g_{lj}\hat{g}^{jk}\hat{g}^{il} = \frac{1}{(\lambda_{p1})^4} + \frac{1}{(\lambda_{p2})^4} + \frac{1}{(\lambda_{p3})^4} \quad (28)$$

$$L_3 = g^{mi}g^{nj}g^{ok}\hat{g}_{jm}\hat{g}_{kn}\hat{g}_{io} = (\lambda_{p1})^6 + (\lambda_{p2})^6 + (\lambda_{p3})^6$$
$$L^3 = g_{mi}g_{nj}g_{ok}\hat{g}^{jm}\hat{g}^{kn}\hat{g}^{io} = \frac{1}{(\lambda_{p1})^6} + \frac{1}{(\lambda_{p2})^6} + \frac{1}{(\lambda_{p3})^6} \quad (29)$$

These invariants are characterized by having no coupling terms in the principal stretches and only involving the principal stretches to even powers. In general we can construct invariants L_n and L^n of this form with principal stretches to any even power $2n$ such that

$$L_n = (\lambda_{p1})^{2n} + (\lambda_{p2})^{2n} + (\lambda_{p3})^{2n}$$
$$L^n = (\lambda_{p1})^{-2n} + (\lambda_{p2})^{-2n} + (\lambda_{p3})^{-2n} \quad (30)$$

(n is used as an index in the left hand side of the above formulas).

7. Stress tensors

The deformation of the continuum associated with the movement of point P to $\hat{P}$ is assumed to be caused by the action of forces consisting of both body forces and forces applied at the boundary surfaces. At the point $\hat{P}$, we can visualize an infinitesimal parallelepiped with sides corresponding to the vectors $\hat{\mathbf{g}}_1\,\mathrm{d}\theta^1$, $\hat{\mathbf{g}}_2\,\mathrm{d}\theta^2$ and $\hat{\mathbf{g}}_3\,\mathrm{d}\theta^3$. A force vector at $\hat{P}$ denoted by $\mathrm{d}\mathbf{F}$, can be written in the form:

$$\mathrm{d}\mathbf{F} = \mathrm{d}\widehat{F}^j\hat{\mathbf{g}}_j = \mathrm{d}F^j\mathbf{g}_j = \mathrm{d}\widehat{\mathbf{T}}^i\,\mathrm{d}\hat{A}_i = \mathrm{d}\mathbf{T}^i\,\mathrm{d}A_i = \mathrm{d}\overline{\mathbf{T}}^i\,\mathrm{d}\bar{A}_i \quad (31)$$

with $\mathrm{d}\widehat{F}^j$ and $\mathrm{d}F^j$ being the contravariant force vector components with respect to base vectors in the deformed and undeformed configuration, respectively, $\mathrm{d}\widehat{\mathbf{T}}^i$, $\mathrm{d}\overline{\mathbf{T}}^i$ and $\mathrm{d}\mathbf{T}^i$ are stress vectors acting on the faces of the infinitesimal parallelepiped at point $\hat{P}$ with respect to base vectors in the deformed and undeformed configuration, and $\mathrm{d}\hat{A}_i$, $\mathrm{d}A_i$ and $\mathrm{d}\bar{A}_i$ are the area vectors components defined by

$$\mathrm{d}\hat{\mathbf{A}} = \mathrm{d}\hat{A}_i\hat{\mathbf{g}}^i = \mathrm{d}\bar{A}_i\mathbf{g}^i \qquad \mathrm{d}\mathbf{A} = \mathrm{d}A_i\mathbf{g}^i \tag{32}$$

Table 1
Summary of stress tensor quantities

	Stress tensor	Physical counterpart	Physical stress for initial cartesian system
Eulerian	τ^{ij}	$\varsigma^{ij} = \tau^{ij}\dfrac{\sqrt{\hat{g}_{(jj)}}}{\sqrt{\hat{g}^{(ii)}}}$	$\varsigma^{ij} = \tau^{ij}\dfrac{\sqrt{\hat{g}_{(jj)}}}{\sqrt{\hat{g}^{(ii)}}}$
Cauchy	σ^{ij}	$\sigma^{ij}\dfrac{\sqrt{g_{(jj)}}}{\sqrt{g^{(ii)}}}$	σ^{ij}
Second Piola–Kirchhoff	π^{ij}	$s^{ij} = \pi^{ij}\lambda_{(j)}\dfrac{\sqrt{g_{(jj)}}}{\sqrt{g^{(ii)}}}$	$s^{ij} = \pi^{ij}\lambda_{(j)}$
First Piola–Kirchhoff	t^{ij}	$f^{ij} = t^{ij}\dfrac{\sqrt{g_{(jj)}}}{\sqrt{g^{(ii)}}}$	$f^{ij} = t^{ij}$

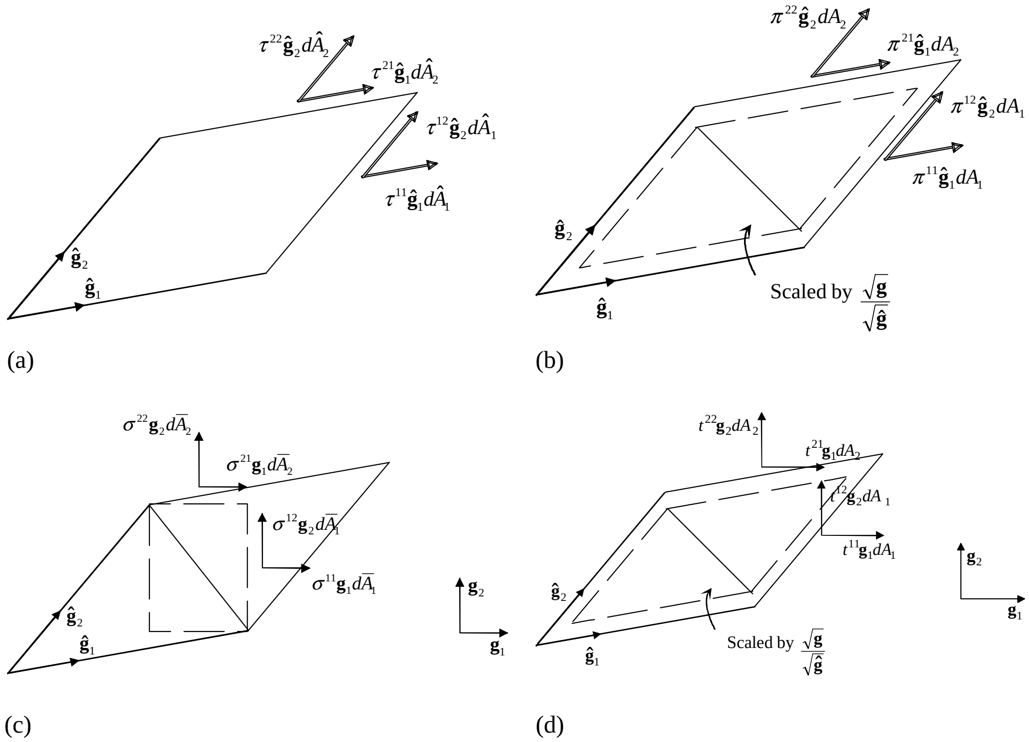

Fig. 4. Common stress tensors: (a) Eulerian stress tensor, (b) second Piola–Kirchhoff stress tensor, (c) Cauchy stress tensor and (d) first Piola–Kirchhoff stress tensor.

where $\mathrm{d}\hat{\mathbf{A}}$ and $\mathrm{d}\mathbf{A}$ are area vectors in the deformed and undeformed state, respectively (note: $\mathrm{d}\hat{A}_i = J\,\mathrm{d}A_i$ and $\mathrm{d}\bar{A}_i = \overline{G}_i^j\,\mathrm{d}\hat{A}_j$).

The force vector components can be equilibrated by various systems of stresses referred to base vectors either in the deformed or undeformed body. Some of the common stress tensors are defined by

$$\mathrm{d}\mathbf{F} = \tau^{ij}\hat{\mathbf{g}}_j\,\mathrm{d}\hat{A}_i = \pi^{ij}\hat{\mathbf{g}}_j\,\mathrm{d}A_i = t^{ij}\mathbf{g}_j\,\mathrm{d}A_i = \sigma^{ij}\mathbf{g}_j\,\mathrm{d}\bar{A}_i \tag{33}$$

where τ^{ij} is the Eulerian stress tensor, π^{ij} is the second Piola–Kirchhoff stress tensor, t^{ij} is the first Piola–Kirchhoff stress tensor and σ^{ij} is the Cauchy stress tensor. The Eulerain and the second Piola–Kirchhoff stress tensors are referred to oblique axes that are aligned with the tangent base vectors $\hat{\mathbf{g}}_j$ at $\hat{P}$ in the deformed state. Both the Cauchy stress tensor and the first Piola–Kirchhoff stress tensor are aligned with the directions of the initial tangent base vectors in the undeformed state. The Eulerian stress tensor, the second Piola–Kirchhoff stress tensor and the Cauchy stress tensor are all symmetric. The stress tensors are related by

$$\tau^{ij}J = \pi^{ij} \qquad \sigma^{ij} = G_m^i G_n^j \tau^{mn} \qquad t^{ij} = \pi^{ir}G_r^j \tag{34}$$

The stress tensors do not necessarily have units of force per unit area and physical counterparts can be derived and are listed in Table 1. Fig. 4 shows a two-dimensional representation of the common stress tensors.

8. Virtual work and conjugate deformations

In this section we will derive the part of the virtual work associated with the work of the stresses under a variation of the displacement field. As discussed in Renton (1987), the variation of work of a continuum associated with a variation in the displacement field, is the scalar product between the variation of the displacements between opposite faces of the deformed parallelepiped and the associated forces. That is

$$\delta(\mathrm{d}W) = \mathrm{d}\widehat{\mathbf{T}}^i\,\mathrm{d}\hat{A}_{(i)}\cdot\delta\hat{\mathbf{g}}_i\,\mathrm{d}\theta^{(i)} = \mathrm{d}\widehat{\mathbf{T}}^i\cdot\delta\hat{\mathbf{g}}_i\,\mathrm{d}\widehat{V} = \mathrm{d}\mathbf{T}^i\cdot\delta\hat{\mathbf{g}}_i\,\mathrm{d}V \tag{35}$$

No restrictions in terms of boundary or continuity conditions will be placed on the variation of the displacements. Traditionally the virtual work equation is derived by constructing the dot product of the equilibrium equations with a variation in the displacement vector and integrating over the volume. This is a much more involved procedure which results in the same equations. The virtual work equations are a statement of equilibrium and can be used to assess conjugate pairs of stress and strain or deformation measures. In terms of the Eulerian stress tensor we can write:

$$\delta(\mathrm{d}W) = \mathrm{d}\widehat{\mathbf{T}}^i\cdot\delta\hat{\mathbf{g}}_i\,\mathrm{d}\widehat{V} = \tau^{ij}\hat{\mathbf{g}}_j\cdot\delta\hat{\mathbf{g}}_i\,\mathrm{d}\widehat{V} \tag{36}$$

Using the symmetry condition for the Eulerian stress tensor, the above equation can be rewritten as

$$\delta(\mathrm{d}W) = \tfrac{1}{2}\tau^{ij}\,\delta(\hat{\mathbf{g}}_j\cdot\hat{\mathbf{g}}_i)\,\mathrm{d}\widehat{V} = \tau^{ij}\,\delta\gamma_{ij}\,\mathrm{d}\widehat{V} \tag{37}$$

The Eulerian stress tensor is conjugate to the strain tensor γ_{ij}. Here the variation of the strain tensor is subjected to the same continuity and boundary conditions as $\tau^{ij}\hat{\mathbf{g}}_j$. Substituting the relationship between the Eulerian stress tensor and the physical counterpart (also sometimes referred to as the "true stress", see Treloar, 1975), Eq. (37) becomes:

$$\delta(\mathrm{d}W) = \left(\varsigma^{ij}\sqrt{\hat{g}^{(ii)}}\right)\frac{\hat{\mathbf{g}}_j}{\sqrt{\hat{g}_{(jj)}}}\cdot\delta\hat{\mathbf{g}}_i\,\mathrm{d}\widehat{V} \tag{38}$$

Recall the definition for the relative stretch:

$$\lambda_{(i)} = \frac{\sqrt{\hat{g}_{(ii)}}}{\sqrt{g_{(ii)}}} = \frac{\sqrt{\hat{\mathbf{g}}_i \cdot \hat{\mathbf{g}}_{(i)}}}{\sqrt{g_{(ii)}}} \tag{39}$$

Taking the first variation of the relative stretch yields:

$$\delta\lambda_{(i)} = \frac{\hat{\mathbf{g}}_i \cdot \delta\hat{\mathbf{g}}_{(i)}}{\sqrt{\hat{g}_{(ii)}}\sqrt{g_{(ii)}}} \tag{40}$$

If we restrict our attention to principal Eulerian physical stresses which align with the directions of principal stretches then using the above and the fact that for the principal stretch direction $\sqrt{\hat{g}^{(ii)}} = 1/\lambda_{pi}$ we can write:

$$\delta(\mathrm{d}W) = \varsigma_p^{i(i)} \frac{\delta\lambda_{pi}}{\lambda_{p(i)}} \mathrm{d}\widehat{V} = \varsigma_p^{i(i)}\, \delta \ln(\lambda_{pi})\, \mathrm{d}\widehat{V} \tag{41}$$

We find that the principal Eulerian physical stresses are conjugate to the natural log of the principal stretch when the principal stress and stretch directions are aligned.

Since the Cauchy stress tensor and the Almansi strain tensor are both coordinate transformations of the Eulerian stress and strain tensor, respectively, we have

$$\delta(\mathrm{d}W) = \mathrm{d}\overline{\mathbf{T}}^i\, \mathrm{d}\bar{A}_{(i)} \cdot \delta\mathbf{g}_i\, \mathrm{d}\bar{\theta}^{(i)} = \sigma^{ij}\mathbf{g}_j \cdot \delta\mathbf{g}_i\, \mathrm{d}\widehat{V} = \sigma^{ij}\, \delta(\bar{\gamma}_{ij})\, \mathrm{d}\widehat{V} \tag{42}$$

The variation has been taken with respect to the initial tangent base vectors that is $\delta\hat{\mathbf{g}}_i\, \mathrm{d}\theta^i = \delta\mathbf{g}_j G_i^j\, \mathrm{d}\theta^i = \delta\mathbf{g}_i\, \mathrm{d}\bar{\theta}^i$. The Cauchy stress tensor is conjugate to the Almansi strain tensor. The variation in work for the Cauchy stress tensor can also be written in the form:

$$\delta(\mathrm{d}W) = \tfrac{1}{2}\overline{G}_m^i\, \overline{G}_n^j \sigma^{mn} \cdot \delta\hat{\mathbf{g}}_{ij}\, \mathrm{d}\widehat{V} \tag{43}$$

For the variation in work associated with the second Piola–Kirchhoff stress tensor we have

$$\delta(\mathrm{d}W) = \mathrm{d}\mathbf{T}^i \cdot \delta\hat{\mathbf{g}}_i\, \mathrm{d}V = \tfrac{1}{2}\pi^{ij}\delta(\hat{\mathbf{g}}_j \cdot \hat{\mathbf{g}}_i)\, \mathrm{d}V = \pi^{ij}\delta\gamma_{ij}\, \mathrm{d}V \tag{44}$$

The second Piola–Kirchhoff stress tensor like the Eulerian stress tensor is also conjugate to the strain tensor γ_{ij}. This is because the second Piola–Kirchhoff stress tensor is equal to the Eulerian stress tensor scaled by the invariant J. Substituting the relationship between the second Piola–Kirchhoff stress tensor and the physical counterpart, Eq. (44) becomes:

$$\delta(\mathrm{d}W) = \left(s^{ij}\sqrt{g^{(ii)}}\right) \frac{\hat{\mathbf{g}}_j}{\sqrt{\hat{g}_{(jj)}}} \cdot \delta\hat{\mathbf{g}}_i\, \mathrm{d}V \tag{45}$$

As with the physical Eulerian stresses, let us restrict our attention to a state of principal stresses. In this case, the associated stretches do not have to be principal. Using Eq. (13), Eq. (45) becomes:

$$\delta(\mathrm{d}W) = s_p^{i(i)}\, \delta\lambda_i\, \mathrm{d}V \tag{46}$$

The principal Lagrangian physical stresses are conjugate to their associated stretches. When the stretches are also principal then we have the following:

$$\delta(\mathrm{d}W) = s_p^{i(i)}\, \delta\lambda_{pi}\, \mathrm{d}V \tag{47}$$

For the variation in work associated with the first Piola–Kirchhoff stress tensor we have

$$\delta(\mathrm{d}W) = \mathrm{d}\mathbf{T}^i\, \mathrm{d}A_{(i)} \cdot \delta\hat{\mathbf{g}}_i\, \mathrm{d}\theta^{(i)} = t^{ij}\mathbf{g}_j \cdot \delta\hat{\mathbf{g}}_i\, \mathrm{d}V = t^{ij}\delta(\mathbf{g}_j \cdot \hat{\mathbf{g}}_i)\, \mathrm{d}V \tag{48}$$

The first Piola–Kirchhoff stress tensor is conjugate to the deformation measure $\mathbf{g}_j \cdot \hat{\mathbf{g}}_i$ which represents the dot product of the tangent base vectors with the initial tangent base vectors. Substituting the relationship between the first Piola–Kirchhoff stress tensor and its physical counterpart, results in

$$\delta(\mathrm{d}W) = (f^{ij}\sqrt{g^{(ii)}})\frac{\mathbf{g}_j}{\sqrt{g_{(jj)}}} \cdot \delta\hat{\mathbf{g}}_i \,\mathrm{d}V \tag{49}$$

For an initial Cartesian coordinate system, the first Piola–Kirchhoff stress tensor is equal to its physical counterpart and conjugate to the projections of the associated tangent base vectors onto the initial tangent base vectors.

9. Isotropic hyperelastic strain energy density

The strain energy density $\mathrm{d}U$ with respect to the initial volume is related to the variation in work by the equation:

$$\delta(\mathrm{d}U)\,\mathrm{d}V = \delta(\mathrm{d}W) = \mathbf{dT}^i \cdot \delta\hat{\mathbf{g}}_i \,\mathrm{d}V \tag{50}$$

For a hyperelastic material, the stress tensors can be derived from the strain energy and we can therefore conclude from Eqs. (50), (37), and (34) the following:

$$\pi^{ij} = \tau^{ij}J = 2\frac{\partial \mathrm{d}U}{\partial \hat{g}_{ij}} \quad J\sigma^{ij} = 2G^i_m G^j_n \frac{\partial \mathrm{d}U}{\partial \hat{g}_{mn}} \quad t^{ij} = 2G^j_k \frac{\partial \mathrm{d}U}{\partial \hat{g}_{ik}} \tag{51}$$

and for the principal physical Lagrangian and Eulerian stresses

$$s_p^{(ii)} = \frac{\partial \mathrm{d}U}{\partial \lambda_{pi}} \quad \varsigma_p^{(ii)} = \frac{\lambda_{p(i)}}{J}\frac{\partial \mathrm{d}U}{\partial \lambda_{pi}} \tag{52}$$

Ogden (1997) and Treloar (1975) present a detailed discussion about the restrictions on the form of the strain energy density. Here we make several postulates as to conditions to which the strain energy density must hold and use this to propose the form of the strain energy density. These postulates include:

1. The strain energy density must be non-negative for all deformations.
2. The strain energy density must be invariant under coordinate transformation.
3. The strain energy must be a function of either the stretch or strain invariants and because of isotropy be symmetrical with respect to the principal stretches λ_{p1}, λ_{p2} and λ_{p3}.
4. The strain energy density must have a zero value at the undeformed state ($\lambda_{p1} = 1$, $\lambda_{p2} = 1$ and $\lambda_{p3} = 1$).
5. The strain energy density must be a minimum at the undeformed state. This guarantees that the material is stress free at the undeformed state. Hence

$$\left(\frac{\partial \mathrm{d}U}{\partial \lambda_{pi}}\right)_{\text{undeformed state}} = 0 \quad i = 1,2,3 \tag{53}$$

$$\left(\frac{\partial^2 \mathrm{d}U}{\partial \lambda_{pi}^2}\right)_{\text{undeformed state}} > 0 \quad i = 1,2,3 \tag{54}$$

6. The strain energy density must approach positive infinity at a singularity ($\lambda_{p1} = 0$ or $\lambda_{p2} = 0$ or $\lambda_{p3} = 0$) and for very large deformations ($\lambda_{p1} = \infty$ or $\lambda_{p2} = \infty$ or $\lambda_{p3} = \infty$). Stresses, on the other hand,

should approach negative infinity at a singularity and positive infinity for very large deformations. This has some experimental confirmation from tests on rubber in compression and tension.

7. The strain energy density is assumed to be decomposed into two components, one $\mathrm{d}U_{\mathrm{incomp}}$ associated with incompressibility or the strain energy density under constrained volume change or volume constant distortion. This component therefore involves no coupling of the principal stretches hence because of postulate 3, must be a linear function of the general invariants L_n and L^n. This would also satisfy postulate 6 since the invariants L_n and L^n contain even powers of the principal stretches and the reciprocals of the principal stretches. Therefore

$$\begin{aligned}\mathrm{d}U_{\mathrm{incomp}} &= \sum_{n=1}^{r} \frac{A_n}{2n}(L_n - 3) + \frac{B_n}{2n}(L^n - 3)\,\mathrm{d}V \\ &= \sum_{n=1}^{r} \frac{A_n}{2n}[(\lambda_{p1})^{2n} + (\lambda_{p2})^{2n} + (\lambda_{p3})^{2n} - 3] + \frac{B_n}{2n}[(\lambda_{p1})^{-2n} + (\lambda_{p2})^{-2n} + (\lambda_{p3})^{-2n} - 3]\,\mathrm{d}V\end{aligned} \tag{55}$$

where A_n and B_n are material constants, and r is the termination point of the summation. The expression proposed above for $\mathrm{d}U_{\mathrm{incomp}}$ is the same as the general expression presented by Mooney (1940) for incompressible isotropic materials and was based on the assumption that the traction in simple shear is an analytical function of the shear. The expression proposed above for $\mathrm{d}U_{\mathrm{incomp}}$ also satisfies the Valanis–Landel hypothesis for the strain energy density for incompressible isotropic materials. Their hypothesis stated that the strain energy density for incompressible isotropic materials should be capable of representation as the sum of three separate but identical functions of each of the individual principal stretches (see Ogden, 1997; Treloar, 1975). The Mooney constants would correspond to A_1 and B_1.

The other component of the strain energy density $\mathrm{d}U_{\mathrm{comp}}$ is associated with the compressibility or specific volume change as referred to in Freudenthal (1966). Because of isotropy $\mathrm{d}U_{\mathrm{comp}}$ must be a function of the volumetric dilation through the invariant J. Hence

$$\mathrm{d}U = \mathrm{d}U_{\mathrm{incomp}}(L_n, L^n) + \mathrm{d}U_{\mathrm{comp}}(J) \tag{56}$$

8. The compressibility component of the strain energy density is further split into two terms of the form $\mathrm{d}U_{\mathrm{comp}}(J) = U(J) - (\sum_{n=1}^{r} A_n - B_n)\ln J$. The logarithmic term is needed (provided $A_n \neq B_n$) so that, the material is stress free at the undeformed state. This logarithm term is associated with a hydrostatic pressure needed to maintain the stress free state at the undeformed configuration.

9. In agreement with postulate 6 and as discussed in Simo and Pister (1984), the strain energy density component $\mathrm{d}U_{\mathrm{comp}}(J)$ should approach infinity at both a singularity $J \to 0$ and infinite volume change $J \to \infty$. Simo and Pister (1984) proposed $\mathrm{d}U_{\mathrm{comp}}(J) = \frac{1}{2}\Lambda(\ln J)^2 - G\ln J$ for neo-Hookean isotropic elasticity as one possibility, where G is the shear modulus and Λ the Lamé constant. Other forms for $\mathrm{d}U_{\mathrm{comp}}(J)$ are discussed in Cescotto and Fonder (1979), Häggblad and Sundberg (1983), Brink and Stein (1996) and Rüter and Stein (2000).

Ehlers and Eipper (1998) who examined the lateral strain under uniaxial loading observed that problems (unphysical results) arise from elasticity laws that proceed from an additive split of the strain energy into pure isochoric and pure volumetric parts. Plots of numerically obtained longitudinal strain versus lateral strain showed unrealistic results for longitudinal strains approaching −1 ($J \to 0$). The Simo and Pister (1984) proposal investigated by Ehlers and Eipper (1998) did not exhibit unrealistic results because of the logarithmic squared term $1/2\Lambda(\ln J)^2$.

As a consequence of postulate 7, it can be shown that for an isotropic material the tangential component of the stress vector is not a function of volumetric dilation. Any volumetric dilation only influences the normal component of the stress vectors. To see this, consider a stress vector $\mathrm{d}\mathbf{T}^i$ acting on the ith surface of the deformed parallelepiped. The stress vector can be split into a normal component and a tangential or

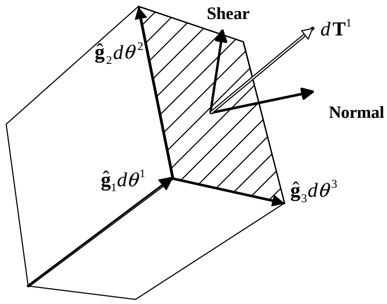

Fig. 5. Shear and normal components of the stress vector.

shear component acting within the plane of the surface. This is depicted in Fig. 5. The projection of the stress vector onto the surface on which it acts is represented by

$$\mathrm{d}\mathbf{T}^i \cdot \hat{\mathbf{g}}_k = \pi^{ij}\hat{g}_{jk} = 2\left(\frac{\partial\,\mathrm{d}U_{\mathrm{incomp}}(L_n, L^n)}{\partial\hat{g}_{ij}} + \frac{\partial\,\mathrm{d}U_{\mathrm{comp}}(J)}{\partial\hat{g}_{ij}}\right)\cdot\hat{g}_{jk} \quad (i \neq k) \tag{57}$$

The portion due to the compressibility component of the strain energy density is then

$$2\frac{\partial\,\mathrm{d}U_{\mathrm{comp}}(J)}{\partial\hat{g}_{ij}}\cdot\hat{g}_{jk} \quad (i \neq k) \tag{58}$$

Because $\mathrm{d}U_{\mathrm{comp}}$ is a function of J only, Eq. (58) must be zero, since from Eq. (7) we have

$$2\frac{\partial\,\mathrm{d}U_{\mathrm{comp}}(J)}{\partial\hat{g}_{ij}}\cdot\hat{g}_{jk} = \frac{\partial\,\mathrm{d}U_{\mathrm{comp}}(J)}{\partial J}J\hat{g}^{ij}\cdot\hat{g}_{jk} = \frac{\partial\,\mathrm{d}U_{\mathrm{comp}}(J)}{\partial J}J\delta^i_k = 0 \quad (i \neq k) \tag{59}$$

This is just a statement of the fact that a hydrostatic pressure will only produce normal stresses on any surface in the deformed configuration. A hydrostatic pressure will not produce shear on any surface of the deformed parallelepiped irrespective of the coordinate configuration.

Based on all the postulates the following strain energy density is proposed for isotropic higher order elasticity:

$$\begin{aligned}
\mathrm{d}U &= \mathrm{d}U_{\mathrm{incomp}} + \mathrm{d}U_{\mathrm{comp}} \\
\mathrm{d}U_{\mathrm{incomp}} &= \sum_{n=1}^{r}\frac{A_n}{2n}(L_n - 3) + \frac{B_n}{2n}(L^n - 3)\,\mathrm{d}V \\
&= \sum_{n=1}^{r}\frac{A_n}{2n}[(\lambda_{p1})^{2n} + (\lambda_{p2})^{2n} + (\lambda_{p3})^{2n} - 3] + \frac{B_n}{2n}[(\lambda_{p1})^{-2n} + (\lambda_{p2})^{-2n} + (\lambda_{p3})^{-2n} - 3]\,\mathrm{d}V \\
\mathrm{d}U_{\mathrm{comp}} &= \sum_{n=1}^{s}\frac{C_n}{2n}(\ln J)^{2n} - \left(\sum_{n=1}^{r}A_n - B_n\right)\ln J\,\mathrm{d}V
\end{aligned} \tag{60}$$

where C_n are material constants, and s is a termination point. The $\mathrm{d}U_{\mathrm{comp}}(J)$ terms are a generalisation of the Simo and Pister (1984) proposal. The $(\sum_{n=1}^{r}A_n - B_n)\ln J$ term has been included in $\mathrm{d}U_{\mathrm{comp}}(J)$ but one could argue that it should be included in $\mathrm{d}U_{\mathrm{incomp}}$ since $(\sum_{n=1}^{r}A_n - B_n)\ln J = (\sum_{n=1}^{r}A_n - B_n)$ $(\ln\lambda_{p1} + \ln\lambda_{p2} + \ln\lambda_{p3})$ which is the sum of three separate but identical functions of each of the individual principal stretches.

For the strain energy density to be a minimum at the undeformed state, the material constants must satisfy the following:

$$\left(\frac{\partial^2 \mathrm{d}U}{\partial\lambda_{pi}^2}\right)_{\text{undeformed state}} = \sum_{n=1}^{r} 2n(A_n + B_n) + C_1 > 0 \tag{61}$$

If the material under consideration is Hookean at infinitesimal strain, the material constants are related to the shear modulus G, bulk modulus K and the Lamé constant Λ by

$$\begin{aligned} G &= \sum_{n=1}^{r} n(A_n + B_n) \qquad \Lambda = C_1 \\ K &= C_1 + \frac{2}{3}\sum_{n=1}^{r} n(A_n + B_n) \end{aligned} \tag{62}$$

Using Eqs. (52) and (60), general expressions for the principal physical Lagrangian and Eulerian stresses can be derived.

$$\begin{aligned} s_p^{(ii)} &= \sum_{n=1}^{r} A_n\left(\lambda_{pi}^{2n-1} - \frac{1}{\lambda_{pi}}\right) - B_n\left(\lambda_{pi}^{-2n-1} - \frac{1}{\lambda_{pi}}\right) + \frac{J}{\lambda_{pi}}p_v \\ &= \sum_{n=1}^{r} A_n\lambda_{pi}^{2n-1} - B_n\lambda_{pi}^{-2n-1} + \frac{J}{\lambda_{pi}}p_{\mathrm{T}} \\ \varsigma_p^{(ii)} &= \frac{1}{J}\sum_{n=1}^{r} A_n(\lambda_{pi}^{2n} - 1) - B_n(\lambda_{pi}^{-2n} - 1) + p_v \\ &= \frac{1}{J}\sum_{n=1}^{r} A_n\lambda_{pi}^{2n} - B_n\lambda_{pi}^{-2n} + p_{\mathrm{T}} \end{aligned} \tag{63}$$

where p_{T} is the total hydrostatic pressure component of the principal stresses defined by

$$p_{\mathrm{T}} = \sum_{n=1}^{s} \frac{C_n}{J}(\ln J)^{2n-1} - \sum_{n=1}^{r} \frac{A_n - B_n}{J} = p_v + p_o \tag{64}$$

where $p_o = -\sum_{n=1}^{r}(A_n - B_n)/J$ and $p_v = \sum_{n=1}^{s}(C_n/J)(\ln J)^{2n-1}$ is the hydrostatic pressure associated with volumetric dilation.

To demonstrate the applicability of the proposed strain energy density expression, several problems involving incompressible and compressible materials will be discussed.

10. Incompressible isotropic hyperelastic material

An incompressible material such as rubber is isometric in that there is no change in volume during deformation and the constraint $J = \lambda_{p1}\lambda_{p2}\lambda_{p2} = 1$ is applied to the strain energy density. Any hydrostatic pressure will do no work since the volume is constrained. The $\mathrm{d}U_{\mathrm{comp}}$ component of the strain energy density is therefore zero. Several strain energy density expressions have been proposed for rubber and some of these are discussed in Treloar (1975), Ogden (1997), Boyce and Arruda (2000) and Bischoff et al. (2000). A review of the research literature involving rubber will not be presented here. The experimental verification of many of the proposed strain energy density expressions for incompressible materials are usually based on experiments involving a state of pure homogeneous strain and include tests such as pure tension, compression, equi-biaxial tension and pure shear. The datum set of test results referred to by many

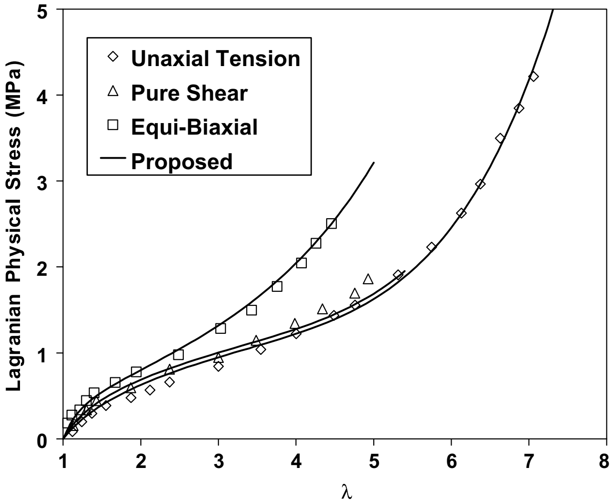

Fig. 6. Comparison with the experimental results of Treloar (1944).

researchers are those of Treloar (1944). A four parameter strain energy density (Eq. (65)) based on Eq. (60) is used here and applied to the problems under pure homogeneous strain as investigated by Treloar (1944).

$$\mathrm{d}U = \tfrac{1}{2}A_1(I_\lambda - 3) + \tfrac{1}{4}A_2(L_2 - 3) + \tfrac{1}{6}A_3(L_3 - 3) + \tfrac{1}{2}B_1(L^1 - 3)\,\mathrm{d}V \tag{65}$$

The experimentally determined shear modulus G was quoted as 0.39 MPa. The material constant A_1 was set to 0.39 MPa and the other material parameters were then estimated using the test results. The material constants which satisfy Eq. (61) so determined are:

$$A_1 = 0.39 \text{ MPa} \qquad A_2 = -0.009615 \text{ MPa}$$

$$A_3 = 0.0002818 \text{ MPa} \qquad B_1 = 0.01267 \text{ MPa}$$

$$G = A_1 + 2A_2 + 3A_3 + B_1 = 0.384 \text{ MPa}$$

$$\left(\frac{\partial^2 \mathrm{d}U}{\partial \lambda_{pi}^2}\right)_{\text{undeformed state}} = 2(A_1 + B_1) + 4A_2 + 6A_3 = 0.7686 > 0$$

Fig. 6 shows a comparison of the test results and those obtained using Eq. (65) with the comparison being good. A much better fit was obtained by Ogden (1997) but involved a six parameter model.

11. Compressible isotropic material under high hydrostatic pressure

To investigate the appropriateness of the proposed form for the compressibility component of the strain energy density it is necessary to look at experiments carried out at very high hydrostatic pressures. In hydrostatic compression tests performed by Adams and Gibson (1930) and Bridgman (1933, 1935, 1945) results were achieved down to values of J of the order of 0.8 (see Ogden, 1997). A three parameter ($A_1 = G$, $C_1 = \Lambda$ and C_2) strain energy density expression for compressible materials is considered here and detailed in the following equation.

$$dU = \tfrac{1}{2}G(I_\lambda - 3) + \tfrac{1}{2}\Lambda(\ln J)^2 + \tfrac{1}{4}C_2(\ln J)^4 - G\ln J\,dV \tag{66}$$

Using Eq. (63), and assuming that under hydrostatic pressure $\lambda_{p1} = \lambda_{p2} = \lambda_{p3} = J^{1/3}$, the general form of the hydrostatic pressure is then given by

$$p = \frac{1}{J}\sum_{n=1}^{r} A_n J^{2n/3} - B_n J^{-2n/3} + p_{\mathrm{T}} \tag{67}$$

which reduces to the following form for the three parameter model considered here:

$$p = G(J^{-1/3} - J^{-1}) + \frac{\Lambda}{J}(\ln J) + \frac{C_2}{J}(\ln J)^3 \tag{68}$$

Based on the experimental results for the compressibility of sodium and N-amyl iodide of Bridgman (1933, 1935), Rubber "A" of Adams and Gibson (1930) and Goodrich D-402 and Koroseal of Bridgman (1945), material parameters were determined and are listed in Table 2. Figs. 7 and 8 show the comparison with the experimental results again showing acceptable comparison.

12. Volume change under uniaxial tension

Penn (1970) measured the volume change of vulcanized natural gum rubber under uniaxial extension. The stress versus stretch data showed the opposite curvature to that demonstrated by the volume change versus stretch data (refer to Figs. 9 and 10). At stretches above 1.5 the "two curves deviate significantly

Table 2
Material parameters

Parameter	Sodium	N-amyl iodide	Rubber "A"	Goodrich D-402	Koroseal
$A_1 = G$	3.333 GPa	0	0	0	0
K	6.3 GPa	1.73 GPa	5.2 GPa	3.55 GPa	2.63 GPa
$C_1 = \Lambda$	4.078 GPa	1.73 GPa	5.2 GPa	3.55 GPa	2.63 GPa
C_2/C_1	11.12	15.87	44.43	48.02	48.02

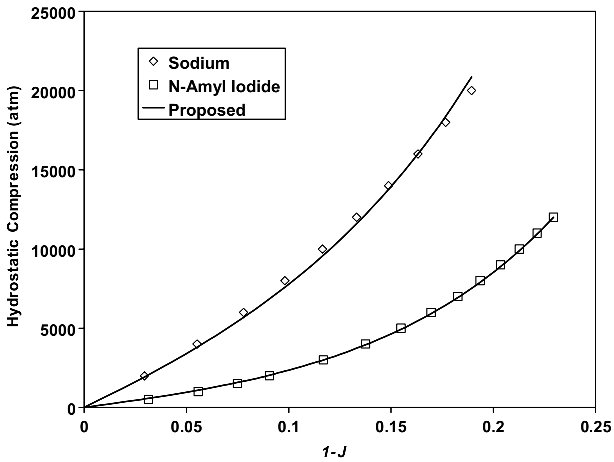

Fig. 7. Comparison with the experimental results of Bridgman (1933, 1935).

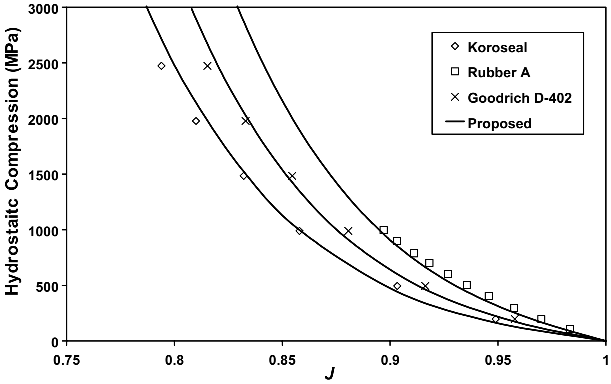

Fig. 8. Comparison with the experimental results of Adams and Gibson (1930) and Bridgman (1945).

when the volume change becomes concave downward while the stress curves upward''. Penn (1970) argued that because of this deviation the strain energy density could not be decomposed into the sum of an incompressible and compressible component.

A four parameter strain energy density was used for this example and is listed below:

$$dU = \tfrac{1}{2}A_1(I_\lambda - 3) + \tfrac{1}{2}B_1(L^1 - 3) + \tfrac{1}{4}A_2(L_2 - 1) + \tfrac{1}{2}C_1(\ln J)^2 - (A_1 + A_2 - B_1)\ln J\, dV \tag{69}$$

Eq. (69) was used to derive expressions for the Lagrangian physical normal and lateral stresses. The parameter C_1 was set to the bulk modulus estimated from Penn (1970) at 2000 MPa. Penn (1970) quoted the Mooney constants as approximately 0.361 and 0.165. These correspond to parameters A_1 and B_1, respectively. The A_1 parameter was set at 0.361 and the remaining two parameters (A_2 and B_1) in the model of Eq. (69) were estimated as

$$A_1 = 0.361\ \text{MPa} \quad B_1 = 0.22\ \text{MPa} \quad A_2 = 0.1\ \text{MPa} \quad C_1 = K = 2000\ \text{MPa}$$
$$G = A_1 + B_1 + 2A_2 = 0.781\ \text{MPa}$$

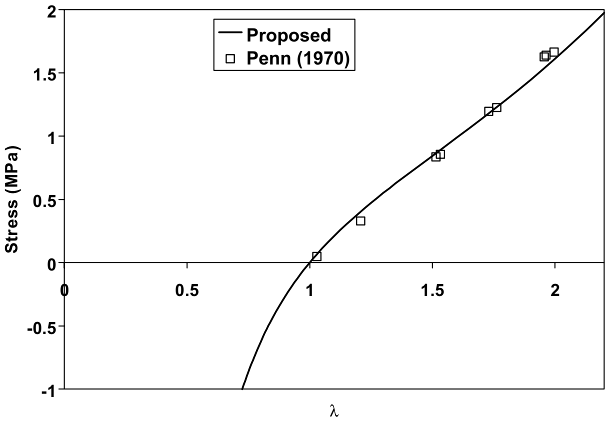

Fig. 9. Comparison of the stretch versus uniaxial stress experimental data of Penn (1970) with proposed model.

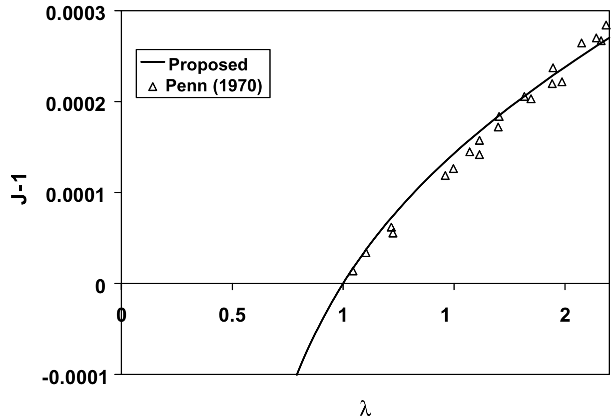

Fig. 10. Comparison of the stretch versus volume change experimental data of Penn (1970) with proposed model.

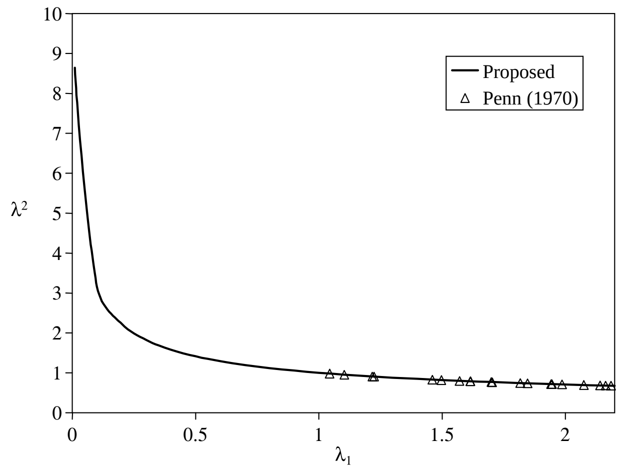

Fig. 11. Lateral stretch λ_2 versus longitudinal stretch λ_1.

For a given value of the stretch, the invariant J was solved to give a zero lateral stress. The normal stress was then calculated for the given stretch and estimated J. Figs. 9 and 10 show a comparison between the model predictions and the experimental data of Penn (1970) showing excellent comparison.

Fig. 11 shows the longitudinal stretch versus the lateral stretch using the experimental example of Penn (1970). The predictions of the model were obtained for stretch (compression) approaching 0 to demonstrate that unrealistic results would not emerge (as can be ascertained from the form of the expression for the lateral stress which is set to zero and solved for the lateral stretch).

13. Compressible isotropic Hookean material

A simple non-negative strain energy density for a compressible isotropic Hookean type material derived from Eqs. (60) and (62) proposed here is

$$\mathrm{d}U = \tfrac{1}{2}G(I_\lambda - 3) + \tfrac{1}{2}\Lambda(\ln J)^2 - G\ln J\,\mathrm{d}V \tag{70}$$

where the material constants are defined by

$$G = \frac{E}{2(1+\mu)} \qquad \Lambda = \frac{E\mu}{(1+\mu)(1-2\mu)} \tag{71}$$

with E being the elastic modulus, G the shear modulus, μ the Poisson's ratio and Λ the Lamé constant. Eq. (70) was originally proposed by Simo and Pister (1984) for compressible Hookean materials. To guarantee that the strain energy density be positive for all deformations, the Poisson's ratio must be less than 0.5 and greater than -1. This is proved by noting that the stationary value of the strain energy density when the stretches are constrained to be positive, is at the zero stress state with zero strain energy density. One can then show that the stationary point is a minimum when $2G + \Lambda$ is positive and definite.

The constitutive law for the principal physical Lagrangian and Eulerian stresses derived from Eqs. (52) and (70) are:

$$\begin{aligned} s_p^{(ii)} &= G\left(\lambda_{pi} - \frac{1}{\lambda_{pi}}\right) + \frac{J}{\lambda_{pi}}p_v \qquad \varsigma_p^{(ii)} = \frac{G}{J}(\lambda_{pi}^2 - 1) + p_v \\ s_p^{(ii)} &= G\lambda_{pi} + \frac{J}{\lambda_{pi}}\left(p_v - \frac{G}{J}\right) \qquad \varsigma_p^{(ii)} = \frac{G\lambda_{pi}^2}{J} + \left(p_v - \frac{G}{J}\right) \end{aligned} \tag{72}$$

where p_v is equal to $\Lambda \ln J/J$. The stress versus stretch relationship is decomposed into two parts: the term multiplied by the shear modulus is associated with volume constant distortion while the second term is a hydrostatic pressure. Because of isotropy, volumetric dilation can only be caused by the application of a hydrostatic pressure p_v.

This form for the constitutive law has the advantage that all the terms are functions of the principal stretches and their reciprocals which have some vector properties as they are associated with the length of the covariant and contravariant tangent base vectors in the deformed state. Eq. (72) ensures that the principal stresses are zero for an unstrained state when the principal stretches are all unity. If a material is compressed such that the principal stretch in one direction approaches zero (a singularity), the associated principal stress and the hydrostatic pressure p_v both approach negative infinity. The constitutive law defined above reduces to the linear elastic Hookean stress strain law used in engineering theory when the strains are very small.

A simple example of the application of Eq. (72) is the case of uniaxial tension or compression where we assume $\lambda_{p2} = \lambda_{p3}$ and $s_p^{22} = s_p^{33} = 0$. From Eq. (72) we can write for the physical principal longitudinal stress and stretch relationship the following involving one material constant.

$$\begin{aligned} p_v &= -\frac{G}{J}(\lambda_{p2}^2 - 1) \\ s_p^{11} &= \frac{G}{\lambda_{p1}}(\lambda_{p1}^2 - \lambda_{p2}^2) \qquad \varsigma_p^{11} = \frac{G}{J}(\lambda_{p1}^2 - \lambda_{p2}^2) \end{aligned} \tag{73}$$

14. Finite strain constitutive relationship

The constitutive law for a compressible isotropic Hookean material will now be derived in terms of the metric tensor in the deformed state. Recall from Eq. (51), that for the stress tensors we have

$$\pi^{ij} = \tau^{ij}J = 2\frac{\partial\,\mathrm{d}U}{\partial\hat{g}_{ij}} \quad J\sigma^{ij} = 2G_m^iG_n^j\frac{\partial\,\mathrm{d}U}{\partial\hat{g}_{mn}} \quad t^{ij} = 2G_k^j\frac{\partial\,\mathrm{d}U}{\partial\hat{g}_{ik}} \tag{74}$$

The strain energy density proposed in Eq. (70) is a function of the invariants I_λ and J.

To determine Eq. (74) we need the partial derivatives for these invariants with respect to the metric tensor in the deformed state $\hat{g}_{ij}$. From the expressions for the invariants, Eqs. (24) and (26), we can write (see Green and Zerna, 1968):

$$\frac{\partial I_\lambda}{\partial \hat{g}_{ij}} = g^{ij} \qquad \frac{\partial \ln J}{\partial \hat{g}_{ij}} = \frac{1}{2}\hat{g}^{ij} \tag{75}$$

Using Eqs. (23), (70), (74), and (75), we therefore have for the Eulerian and the second Piola–Kirchhoff stress tensor:

$$\pi^{ij} = \tau^{ij}J = G(g^{ij} - \hat{g}^{ij}) + \Lambda\hat{g}^{ij}\ln J = 2G\bar{\gamma}^{ij} + J\hat{g}^{ij}p_v = Gg^{ij} + J\hat{g}^{ij}\left(p_v - \frac{G}{J}\right) \tag{76}$$

where $\bar{\gamma}^{ij}$ is the contravariant Almansi strain tensor. The constitutive law for the mixed second Piola–Kirchhoff stress tensor and Eulerian stress tensor where stresses are aligned with the contravariant tangent base vectors can be derived from Eq. (76), that is

$$\pi^i_k = \tau^i_k J = \pi^{ij}\hat{g}_{jk} = Gg^{ij}\hat{g}_{jk} + J\delta^i_k\left(p_v - \frac{G}{J}\right) \tag{77}$$

It is easy to see from Eq. (77) that $(p_v - (G/J))$ is a hydrostatic pressure which is normal to any surface in the deformed configuration. The mean normal stress $\frac{1}{3}\pi^i_i$ and $\frac{1}{3}\tau^i_i$ and the deviatoric stresses $'\pi^i_j$ and $'\tau^i_j$ based on Eq. (77) are therefore

$$\begin{aligned} \tfrac{1}{3}\pi^i_i &= \tfrac{1}{3}\tau^i_i J = \tfrac{1}{3}G(I_\lambda - 3) + Jp_v \\ '\pi^i_j &= '\tau^i_j J = G\left(g^{ik}\hat{g}_{kj} - \tfrac{1}{3}\delta^i_j I_\lambda\right) \end{aligned} \tag{78}$$

The physical Lagrangian and Eulerian stresses based on Eq. (76) can be derived as

$$s^{ij} = J\varsigma^{ij}\frac{\sqrt{\hat{g}^{(ii)}}}{\sqrt{g^{(ii)}}} = \frac{\sqrt{\hat{g}_{(jj)}}}{\sqrt{g^{(ii)}}}[2G\bar{\gamma}^{ij} + J\hat{g}^{ij}p_v] \tag{79}$$

Let us consider an initial Cartesian coordinate system and examine the normal and tangential physical stress components of the physical stress vectors $d\mathbf{T}^i$ (refer to Fig. 12). The physical Lagrangian stresses normal to the surfaces of the deformed parallelepiped are given by

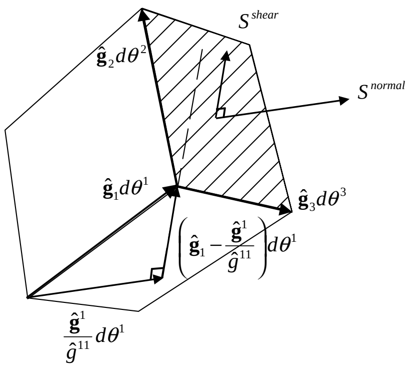

Fig. 12. Deformed parallelepiped showing normal and shear stresses on surface bounded by $\hat{\mathbf{g}}_2\,d\theta^2$ and $\hat{\mathbf{g}}_3\,d\theta^3$.

$$\mathrm{d}\mathbf{T}^i \cdot \frac{\hat{\mathbf{g}}^{(i)}}{\sqrt{\hat{g}^{(ii)}}} = \frac{\pi^{(ii)}}{\sqrt{\hat{g}^{(ii)}}} = \frac{s^{(ii)}}{\sqrt{\hat{g}_{(ii)}}\sqrt{\hat{g}^{(ii)}}} = G\left(\frac{1}{\sqrt{\hat{g}^{(ii)}}} - \sqrt{\hat{g}^{(ii)}}\right) + J\sqrt{\hat{g}^{(ii)}}p_v = G\left(\lambda_{ni} - \frac{1}{\lambda_{ni}}\right) + \frac{J}{\lambda_{ni}}p_v$$
$$= G\lambda_{ni} + \frac{J}{\lambda_{ni}}\left(p_v - \frac{G}{J}\right) \tag{80}$$

where $\lambda_{ni} = 1/\sqrt{\hat{g}^{(ii)}}$ is the normal component of the stretch λ_i (refer to Fig. 13).

The normal physical Lagrangian stress component is a function of the normal component of the stretch λ_{ni} and the volumetric invariant J. Eq. (80) is also true if the initial coordinates are general. Let us now examine the physical Lagrangian shear components of the physical stress vectors that is

$$\mathrm{d}\mathbf{T}^i \cdot \frac{\left(\hat{\mathbf{g}}_{(i)} - \frac{\hat{\mathbf{g}}^{(i)}}{\hat{g}^{(ii)}}\right)}{\sqrt{\hat{g}_{(ii)} - \frac{1}{\hat{g}^{(ii)}}}} = \frac{s^{ij}\hat{g}_j}{\sqrt{\hat{g}_{(jj)}}} \cdot \frac{\left(\hat{\mathbf{g}}_{(i)} - \frac{\hat{\mathbf{g}}^{(i)}}{\hat{g}^{(ii)}}\right)}{\sqrt{\hat{g}_{(ii)} - \frac{1}{\hat{g}^{(ii)}}}} = G\sqrt{\hat{g}_{(ii)} - \frac{1}{\hat{g}^{(ii)}}} = G\lambda_{si} \tag{81}$$

where $\lambda_{si} = \sqrt{\hat{g}_{(ii)} - (1/\hat{g}^{(ii)})}$ is the tangential component of stretch λ_i (refer to Fig. 13).

Unlike Eq. (80), Eq. (81) is only true if the initial coordinates are Cartesian. Importantly, the physical shear stress orthogonal to the normal stress on any of the surfaces of the deformed parallelepiped is not a function of the material constant "Λ" which relates to volumetric dilation. We can arrive at the same equations for the normal and tangential components of the stress vector by looking at the strain energy density which can be expressed in the following form:

$$\mathrm{d}U = \tfrac{1}{2}G(I_\lambda - 3) + \tfrac{1}{2}\Lambda(\ln J)^2 - G\ln J\,\mathrm{d}V$$
$$= \tfrac{1}{2}G([\lambda_{n1}^2 + \lambda_{s1}^2] + \lambda_2^2 + \lambda_3^2 - 3) + \tfrac{1}{2}\Lambda(\ln\{\lambda_{n1}[\hat{g}_{22}\hat{g}_{33} - \hat{g}_{23}\hat{g}_{23}]^{1/2}\})^2$$
$$- G\ln(\lambda_{n1}[\hat{g}_{22}\hat{g}_{33} - \hat{g}_{23}\hat{g}_{23}]^{1/2})\,\mathrm{d}V \tag{82}$$

where $J/\lambda_{n1} = [\hat{g}_{22}\hat{g}_{33} - \hat{g}_{23}\hat{g}_{23}]^{1/2}$ is the ratio of the surface area bound by $\hat{\mathbf{g}}_2\,\mathrm{d}\theta^2$ and $\hat{\mathbf{g}}_3\,\mathrm{d}\theta^3$ in the deformed state to the initial state, $\lambda_{n1} = 1/\sqrt{\hat{g}^{11}}$ and $\lambda_{s1} = \sqrt{\hat{g}_{11} - (1/\hat{g}^{11})}$ are the components of the stretch λ_1 normal and tangential, respectively, to the surface area bounded by $\hat{\mathbf{g}}_2\,\mathrm{d}\theta^2$ and $\hat{\mathbf{g}}_3\,\mathrm{d}\theta^3$ as shown in Fig. 12. The shear component is in the direction defined by the vector $\hat{\mathbf{g}}_1 - (\hat{\mathbf{g}}^1/\hat{g}^{11}) = -(\hat{g}^{12}/\hat{g}^{11})\hat{\mathbf{g}}_2 - (\hat{g}^{13}/\hat{g}^{11})\hat{\mathbf{g}}_3$. The normal S^{normal} and shear S^{shear} physical Lagrangian stresses acting on the surface bound by $\hat{\mathbf{g}}_2\,\mathrm{d}\theta^2$ and $\hat{\mathbf{g}}_3\,\mathrm{d}\theta^3$, can be obtained from the strain energy density, Eq. (82) and are:

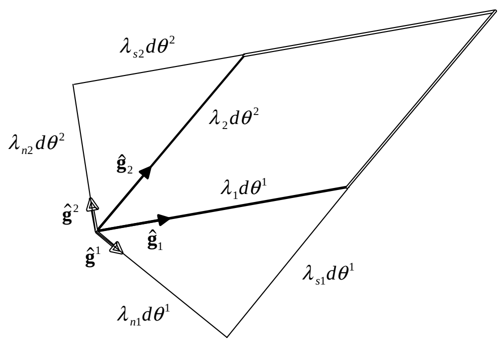

Fig. 13. Two-dimensional deformed parallelogram showing normal and shear components of stretch.

$$S^{\text{normal}} = \frac{\partial \mathrm{d}U}{\partial \lambda_{n1}} = G\left(\lambda_{n1} - \frac{1}{\lambda_{n1}}\right) + \frac{J}{\lambda_{n1}} p_v$$
$$S^{\text{shear}} = \frac{\partial \mathrm{d}U}{\partial \lambda_{s1}} = G\lambda_{s1} \tag{83}$$

agreeing with Eqs. (80) and (81). Fig. 14 shows a two-dimensional representation of the Lagrangian physical stresses based on the proposed constitutive model for isotropic compressible materials.

By contrast, a Hookean constitutive relationship between Green's strain tensor and the second Piola–Kirchhoff stress tensor does not have this property. To demonstrate this, consider a Cartesian coordinate system for simplicity. The Hookean constitutive relationship is given by

$$\pi^{ij} = 2G\gamma_{ij} + \Lambda\delta^i_j\gamma_{mm} \tag{84}$$

(see Wempner, 1981). The components of the projection of the stress vector onto the surface on which it acts is represented by

$$\mathrm{d}\mathbf{T}^i \cdot \frac{\hat{\mathbf{g}}_k}{\sqrt{\hat{g}_{(kk)}}} = \frac{\pi^{ij}\hat{g}_{jk}}{\sqrt{\hat{g}_{(kk)}}} = \frac{2G\gamma_{ij}\hat{g}_{jk} + \Lambda\gamma_{mm}\hat{g}_{ik}}{\sqrt{\hat{g}_{(kk)}}} \quad (i \neq k) \tag{85}$$

We see that there is a term associated with the material constant "Λ". This is not correct as we saw earlier for an isotropic material; the tangential components of the stress vector should not be a function of volumetric dilation.

The constitutive law for the first Piola–Kirchhoff stress tensor and the Cauchy stress tensor are derived from Eqs. (34) and (76), and are expressed by

$$t^{ij} = G(g^{ir}G^j_r - \hat{\mathbf{g}}^i \cdot \mathbf{g}^j) + J\hat{\mathbf{g}}^i \cdot \mathbf{g}^j p_v$$
$$\sigma^{ij} = \frac{G}{J}(G^i_m G^j_n g^{mn} - g^{ij}) + g^{ij} p_v \tag{86}$$

When the initial coordinate system is Cartesian, the Cauchy stress system is orthogonal and the constitutive law simplifies to

$$\sigma^{ij} = \frac{G}{J}(G^i_m G^j_m - \delta^i_j) + \delta^i_j p_v = \frac{G}{J} G^i_m G^j_m + \delta^i_j\left(p_v - \frac{G}{J}\right) \tag{87}$$

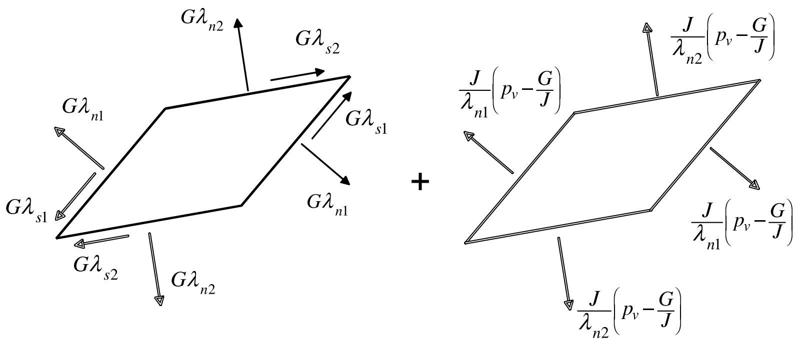

Fig. 14. Two-dimensional deformed parallelogram showing distortion and hydrostatic Lagrangian physical stress components.

15. Simple shear

It is instructive to look at simple shear as an example of the application of the constitutive law defined in Eq. (70). Consider a bar under simple shear. Fig. 15 shows an element subjected to simple shear such that the area remains constant during deformation ($J = 1$). The stretch of the vertical face is unity while the stretch of the inclined face is given by $\lambda = \sqrt{1 + \Delta^2}$ where Δ is the shear displacement. The position vector in the deformed state can be written as

$$\hat{\mathbf{R}} = x\mathbf{i}_1 + (y + \Delta x)\mathbf{i}_2 + z\mathbf{i}_3 \tag{88}$$

where x, y, z are the coordinates of an initial Cartesian coordinate system. The base vectors in the deformed state are therefore

$$\begin{aligned} &\hat{\mathbf{g}}_1 = \mathbf{i}_1 + \Delta\mathbf{i}_2 \quad \hat{\mathbf{g}}_2 = \mathbf{i}_2 \quad \hat{\mathbf{g}}_3 = \mathbf{i}_3 \\ &\hat{\mathbf{g}}^1 = \mathbf{i}_1 \quad \hat{\mathbf{g}}^2 = -\Delta\mathbf{i}_1 + \mathbf{i}_2 \quad \hat{\mathbf{g}}^3 = \mathbf{i}_3 \end{aligned} \tag{89}$$

giving rise to the following invariants:

$$I_\lambda = \Delta^2 + 3 \quad J = 1 \tag{90}$$

The strain energy density for simple shear based on Eqs. (70) and (90) is therefore

$$\mathrm{d}U = \tfrac{1}{2}G\Delta^2\,\mathrm{d}V \tag{91}$$

The physical Lagrangian, Eulerian and Cauchy stresses based on the proposed constitutive law for an isotropic Hookean material are therefore

$$\begin{aligned} &s^{11} = 0 \quad \varsigma^{11} = 0 \quad \sigma^{11} = 0 \\ &s^{12} = G\Delta \quad \varsigma^{12} = G\Delta \quad \sigma^{12} = G\Delta \\ &s^{22} = -G\Delta^2 \quad \varsigma^{22} = \frac{-G\Delta^2}{\lambda} \quad \sigma^{22} = G\Delta^2 \\ &s^{21} = G\Delta\lambda \quad \varsigma^{21} = G\Delta \quad \sigma^{21} = G\Delta \end{aligned} \tag{92}$$

Fig. 16 shows the physical stresses acting on the deformed element under simple shear. We see that there is no stress normal to the vertical face as the normal component of the stretch λ and the volume both remain unchanged during deformation. Ogden (1997, p. 227) also obtained zero stress/force normal to the vertical

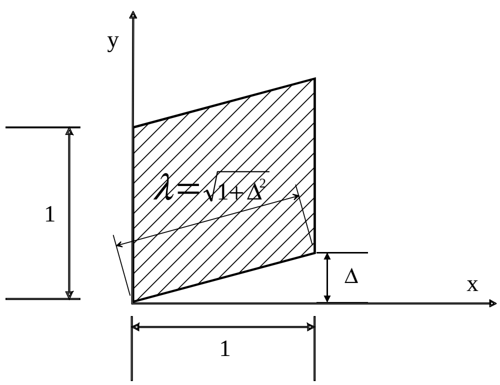

Fig. 15. Simple shear.

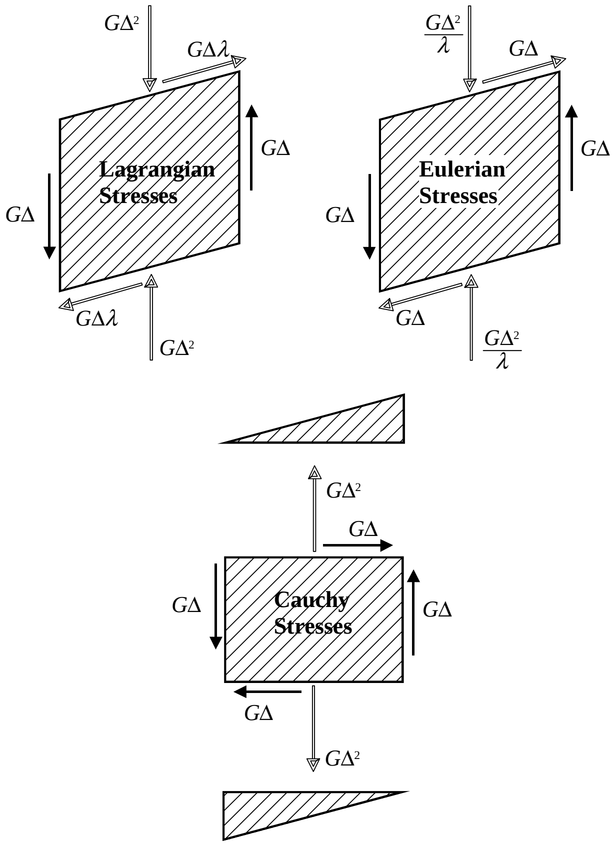

Fig. 16. Simple shear—physical stresses.

face for simple shear. If one uses a Hookean constitutive relationship between Green's strain tensor and the second Piola–Kirchhoff stress tensor as in Eq. (84) and the relationship between the physical Lagrangian stress and its counterpart the second Piola–Kirchhoff stress tensor ($s^{ij} = \pi^{ij}\lambda_{(j)}$), the following is obtained for the physical Lagrangian stresses:

$$s^{11} = \tfrac{1}{2}(2G + \Lambda)\Delta^2\lambda \quad s^{12} = G\Delta \quad s^{22} = \Lambda\Delta^2 \quad s^{21} = G\Delta\lambda \tag{93}$$

Stress s^{11} is not zero and will have a vector component aligned with the vertical face which is a function of Λ. The shear component on the vertical face as argued through this paper should not be a related to the Lamé constant Λ as this is associated with volumetric dilation due to hydrostatic pressure.

The deformation involved in pure torsion of a cylinder is essentially that of simple shear. As discussed in an accompanying paper Attard (2003) many finite strain formulations predict an axial shortening accompanied with a self-equilibrating normal stress of second order. It is shown that the proposed Hookean constitutive relationship does not make that prediction.

16. Conclusions

An endeavour has been made to develop a general expression for the strain energy density for a isotropic hyperelastic material. The strain energy density was decomposed into a incompressible and compressible components. The incompressibility component is the "general" Mooney (1940) expression for higher order elasticity and satisfies the Valanis–Landel hypothesis. The compressibility component of the strain energy density was shown to be a function of the volume invariant J only, and is the strain energy produced by the application of a hydrostatic pressure. The compressibility component proposed is a generalisation of the Simo and Pister (1984) formula for a neo-Hookean material. The compressibility component is associated with volumetric dilation while the incompressibility component is associated with volume constant distortion. The compressibility component of the strain energy density leads to a hydrostatic pressure component of the stress vector which must have no shear component on any surface of the material in any configuration. By contrast, a Hookean constitutive relationship between Green's strain tensor and the second Piola–Kirchhoff stress tensor does not have this property. Comparison with experimental data for the examples involving large deformations of rubber under homogeneous strain, compressible materials under large hydrostatic pressure and measurements of volume changes under uniaxial tension, were good.

The constitutive relationships for a isotropic hyperelastic neo-Hookean material was also derived. The constitutive law for the second Piola–Kirchhoff stress tensor and its physical counterpart are functions of the contravariant Almansi strain tensor and the volumetric invariant J. The neo-Hookean constitutive law for the principal physical Lagrangian and Eulerian stresses consisted of a stretch term and a hydrostatic pressure term.

Acknowledgements

The author would like to thank Professor Serge Cescotto from the University of Liège, Belgium, who read parts of this document and made useful suggestions, and also contributed greatly to the author's understanding of finite strain and stress tensors.

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